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On the development of noise-producing large-scale wavelike eddies in a plane turbulent jet

By LEE-OR MERKINE† AND J. T. C. LIU
Division of Engineering, Brown University, Providence, Rhode Island 02912

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In this paper we study the development of large-scale wavelike eddies in a two-dimensional turbulent jet, extending earlier work on the mixing region (Liu 1974). The basic mean flow develops from one of mixing-region type with an initially specified boundary-layer thickness into a fully developed jet. This study brings out the role of the varicose and sinuous modes as they develop in a growing mean flow. In general, it is found that, for a given frequency parameter, the varicose mode has a shorter streamwise lifetime than the sinuous mode. For lower frequencies, the latter persists past the end of the potential core only to become subject to dissipation by the enhanced fine-scale turbulent activity in that region.

1. Introduction

The basic ideas concerning the elucidation of the development of wavelike eddies in a growing mean turbulent flow were presented previously in Liu (1974), to which we refer readers for an introduction to the formulation of the physical problem. In that paper applications were given for the plane mixing layer with discussion of the near-field properties in relation to observations and control of the development of such eddies. The wavelike eddies ultimately decay and give up their energy to the fine-scale turbulence. In a real jet flow, turbulent diffusion in the fully merged jet region is much more efficient than that in the mixing-layer region. Thus, one of the natural questions raised concerns the role played by this relatively enhanced turbulent diffusion in the development of such eddies. Consideration of their evolution, starting from the mixing layer, should also bring out the relative importance of the varicose and sinuous modes in the near jet noise field. We turn our attention to such geometric applications in this paper, with the aim of gaining insight into the streamwise lifetime or cut-off of the large-scale coherent eddies in a real jet flow. Understanding the mechanisms leading to the cut-off of the noise sources in the jet is of importance concerning the far aerodynamic noise field (Lighthill 1952, 1962; Mollo-Christensen 1960, 1967). In this paper we address ourselves only to the large-scale wavelike eddies, now thought to be the dominant source of jet noise (Bishop, Ffowcs Williams & Smith 1971; Liu 1971, 1974).

† Present address: Department of Meteorology, Massachusetts Institute of Technology Cambridge, Massachusetts 02139.

2. Formulation

The formulation given in Liu (1974) uncouples the wave development from the turbulent mean motion, the argument being that the initial amplitude of the wave responsible for the subsequent development is sufficiently small to make possible the independent calculation of wave development at various frequencies. The real initial amplitudes at the nozzle lip under 'natural' conditions could include contributions from a wide variety of mechanisms such as oscillations of the flow at the jet exit, vibrations of the nozzle wall or noise from the internal flow, the lower bound being the forcing by the turbulent boundary layer on the nozzle wall prior to mixing. In Liu (1974) the latter is used as a basis for estimation of the initial amplitudes, which are *broad banded* in the 'low' frequency spectrum of interest (Kistler & Chen 1963). The calculated subsequent streamwise development of wavelike eddies generates a near field that bears a striking resemblance to observations, both qualitatively and quantitatively. In order to check the influence of the development of wavelike eddies on the mean flow within the same framework, we formulated the fully coupled problem, which includes the effect of the wave or eddy Reynolds stresses on the mean flow development. For the same initial values of the wave energy, the mean motion was found indeed to be negligibly affected. However, we are careful to point out that this statement is only intended to apply to a formulation which treats the wave-induced turbulent Reynolds stresses via an eddy-viscosity model.

It therefore suffices only to mention that we formulated the coupled mean flow–wave interaction model and that its computational results are essentially the same as those of the corresponding uncoupled one in the ranges of initial wave amplitudes of practical interest. Because it is significantly more cumbersome to present the coupled formulation, all our subsequent presentation will be in terms of the simplified version. The main purpose of this paper is to follow the wave development from the mixing region into the merged jet flow and thus to elucidate the effect of the *enhanced* small-scale turbulent 'dissipation' on the kinetic energy of large-scale eddies in the merged region. This then provides insight into the streamwise lifetime or cut-off of the large-scale wavelike eddies in a real jet flow. Since we wish to provide some understanding of the above problem as well as the role played by the sinuous and varicose modes of wave development, we consider the simpler case of a two-dimensional mean flow in which two-dimensional wave motions develop (Liu 1974; Brown & Roshko 1971, 1972).

The mean flow

The configuration of the two-dimensional fully expanded jet is illustrated in figure 1, where $y = y_d$ is the dividing streamline separating fluid particles that originate from within the nozzle from those that originate in the ambient field, u^* is the streamwise component of the velocity along the dividing streamline, δ is the shear-layer width, δ_2 and δ_1 measure the parts of the shear-layer thickness above and below the dividing streamline, respectively, δ_0 is the shear-layer width at the exit of the jet and R is the half-width of the jet exit. The subscripts

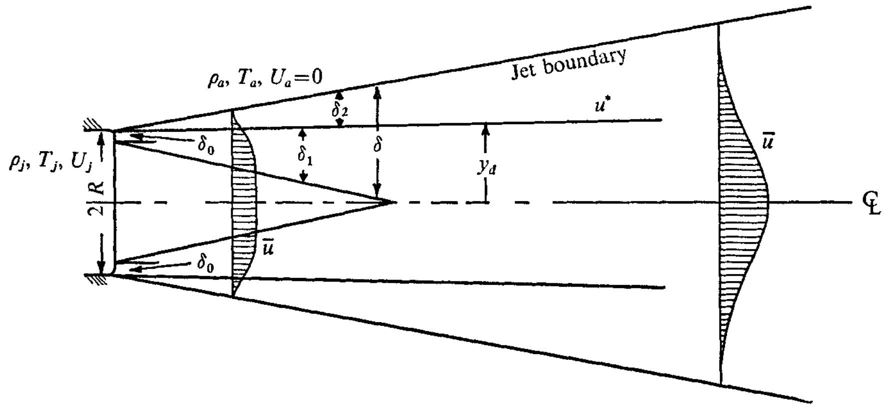


FIGURE 1. Two-dimensional jet flow: schematic.

j and a denote values at the exit of the jet and in the ambient field, respectively, and ρ, T and U are the density, temperature and velocity. The subscript ℄ denotes a value along the centre-line of the jet.

We consider the development of instability waves in a mean flow whose dominant interaction is with the fine-scale turbulence. We invoke Morkovin's (1964) hypothesis that the turbulence structure is unaffected by compressibility and neglect terms which involve turbulent fluctuations in pressure and density. We apply Prandtl's boundary-layer assumptions to the time-averaged equations of motion and assume that the turbulent Prandtl number is unity. We also neglect the unimportant molecular effects. The integrated continuity and momentum equations together with the mechanical energy relation for the mean flow and the thermal energy relation take the following forms for our two-dimensional cold jet:

$$\frac{d}{dx}\int_{-\bar{y}_d}^{\bar{y}_d} \bar{u}\,d\bar{y} = 0, \quad \frac{d}{dx}\int_{-\infty}^{\infty} \bar{u}^2\,d\bar{y} = 0, \qquad (2.1), (2.2)$$

$$\frac{d}{dx}\int_{-\infty}^{\infty} \tfrac{1}{2}\bar{u}^3\,d\bar{y} = -\int_{-\infty}^{\infty} \tilde{\epsilon}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 d\bar{y}, \qquad (2.3)$$

$$\frac{\bar{T}}{T_j} = 1+\tfrac{1}{2}(\gamma-1)\,M_j^2\left[1-\left(\frac{\bar{u}}{U_j}\right)^2\right], \quad M_j = \frac{U_j}{(\gamma R T_j)^{\frac{1}{2}}}, \qquad (2.4)$$

where $\bar{u}$ is the mean streamwise velocity, $\bar{y}$ is the Howarth–Doronitsyn co-ordinate, which is related to the normal co-ordinate y through the relation $d\bar{y} = (\bar{\rho}/\rho_j)\,dy = (T_j/\bar{T})\,dy$, and $\tilde{\epsilon}$ is an x-dependent incompressible eddy viscosity related to the compressible eddy viscosity ϵ via the relation $\bar{\rho}^2\epsilon = \rho_r^2\tilde{\tilde{\epsilon}} \equiv \tilde{\epsilon}$, where ρ_r is a reference density. Alber & Lees (1968) showed that in the mixing region ρ_a is the proper reference density and that $\tilde{\tilde{\epsilon}} = K_\theta U_j \bar{\theta}$, where $\bar{\theta}$ is the transformed momentum thickness and $K_\theta \simeq 0{\cdot}06$. The above expression for $\tilde{\tilde{\epsilon}}$ is inadequate for the description of the merged region of the jet, where diffusion is faster than in the mixing region. Consequently, we choose $\tilde{\tilde{\epsilon}} = 0{\cdot}037 U_{℄}\bar{y}_{\frac{1}{2}}$ as the proper eddy viscosity for the merged region with ρ_j as the reference density; $\bar{y}_{\frac{1}{2}}$ denotes the location at which the velocity is half its centre-line value. (For many possible expressions for the eddy viscosity see Eggers (1966), for example.) It should be pointed out that the eddy viscosity is identically zero outside the shear region.

Next, we normalize the system of equations by referring the physical quantities to their free-stream values at the exit of the jet. Consequently, the velocity, density and temperature are scaled on U_j, ρ_j and T_j, respectively. The transformed half-width $\bar{R}$ of the jet exit is chosen as the reference length scale. In the following analysis we shall deal only with non-dimensional quantities though retaining the same symbols, unless otherwise stated.

Following ideas of Kubota & Dewey (1964) and Alber & Lees (1968) we assign different shape functions to the velocity field above and below the dividing streamline:

$$\bar{u} = \begin{cases} U_{\mathbb{C}}, & 0 \leqslant \bar{y} \leqslant \bar{y}_d - \bar{\delta}_1, \\ U_{\mathbb{C}} - (U_{\mathbb{C}} - u^*)\,[1-(\bar{y}_d-\bar{y})/\bar{\delta}_1]^2, & \bar{y}_d - \bar{\delta}_1 \leqslant \bar{y} \leqslant \bar{y}_d, \\ u^*[1-(\bar{y}-\bar{y}_d)/\bar{\delta}_2]^2, & \bar{y}_d \leqslant \bar{y} \leqslant \bar{y}_d + \bar{\delta}_2, \\ 0, & \bar{y}_d + \bar{\delta}_2 \leqslant \bar{y}, \end{cases} \tag{2.5}$$

$$\bar{\delta}_1 = \frac{U_{\mathbb{C}} - u^*}{U_{\mathbb{C}}}\bar{\delta}, \quad \bar{\delta}_2 = \frac{u^*}{U_{\mathbb{C}}}\bar{\delta}.$$

The last two relations are obtained by matching the shear across the dividing streamline. In the core region $U_{\mathbb{C}} = 1$ and in the developed region $\bar{y}_d = \bar{\delta}_1$. The three unknowns ($u^*, \bar{y}_d$ and $\bar{\delta}$ in the core region and $U_{\mathbb{C}}, u^*$ and $\bar{\delta}$ in the developed region) are determined with the aid of (2.1)–(2.3) subject to the initial conditions that $u^* = 0$, $\bar{y}_d = 1$ and $\bar{\delta} = \bar{\delta}_0$ at $x = 0$.

The mean flow, which is to be used in the local eigenvalue problem subsequently, is best described in terms of the development of u^*, the velocity along the dividing streamline, as a function of the distance downstream. Its behaviour is initially of mixing-region type but it reaches the similar-solution value of about 0·58 prior to merging (Liu 1974). After the end of the potential core, u^* decays and $U_{\mathbb{C}} \sim x^{-\frac{1}{2}}$. The length of the potential core region is found to be $x_c \simeq 25$ while that of the sonic region is $x_s \simeq 48$ for the $M_j = 2{\cdot}22$ jet with an estimated value of δ_0 of 0·10 (note that $R \simeq \bar{R}$). Although no strict comparisons could be made with the round jet, it is mentioned that Eggers (1966) found $x_c \simeq 22$ and $x_s \simeq 50$ for the $M_j = 2{\cdot}22$ round jet.

The wave kinetic energy equation and the eigenvalue problem

Liu (1971, 1974) derived the equation determining the evolution of the wave. In our notation it takes the following form:

$$\frac{d}{dx}\int_{-\infty}^{\infty} \tfrac{1}{2}\bar{u}\,(\overline{u'^2}+\overline{v'^2})\,d\bar{y} = -\int_{-\infty}^{\infty} \frac{1}{\bar{T}}\overline{u'v'}\,\frac{\partial \bar{u}}{\partial \bar{y}}\,d\bar{y} - \int_{-\infty}^{\infty}\left(\bar{T}\overline{u'\frac{\partial p'}{\partial x}} + \overline{v'\frac{\partial p'}{\partial \bar{y}}}\right)d\bar{y}$$
$$-\int_{-\infty}^{\infty} \tilde{\epsilon}\bar{T}^2\left\{\frac{4}{3}\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + \frac{1}{\bar{T}^2}\overline{\left(\frac{\partial v'}{\partial \bar{y}}\right)^2} - \frac{1}{\bar{T}}\overline{\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial \bar{y}}}\right] + \overline{\left(\frac{1}{\bar{T}}\frac{\partial u'}{\partial \bar{y}} + \frac{\partial v'}{\partial x}\right)^2}\right\}d\bar{y}, \tag{2.6}$$

where primes denote components of the large-scale disturbance. Equation (2.6) states that the evolution of the mean kinetic energy of the wave convected by the mean flow is determined by three energy exchange mechanisms: (*a*) transfer of energy from the mean flow to the wave, commonly termed as 'production', the first term on the right-hand side; (*b*) work done by the instability pressure gradients, the second term on the right-hand side; (*c*) energy exchange between the wave and the fine-scale turbulence, which we call 'turbulent dissipation', the last term on the right-hand side. The first two integrals can take either sign depending on the dynamics of the process. (In our case they are positive.) The last integral is always positive since it permits transfer of energy in one direction only: from the wave to the fine-scale turbulence. This result follows the phenomenological assumption that the wave-induced turbulent Reynolds stresses can be related to the wave rates of strain via a postulated eddy viscosity (Liu 1974).

In order to obtain an amplitude equation for each frequency component of the large-scale structure from (2.6), following earlier work (Ko, Kubota & Lees 1970; Liu & Lees 1970; Liu 1974) we assert that the form of any fluctuating component q' of the large-scale structure is given by the eigenfunction $Q\exp(-i\beta t)$ of the *local* linear theory suitably modified by an amplitude function A:

$$q'(x,\eta) = A(x)\,Q(\eta;x)\exp(-i\beta t) + \text{c.c.}, \tag{2.7}$$

where Q is the shape function, $\eta = \bar{y}/\bar{\delta}$ is the local normal co-ordinate, β is a local dimensionless frequency related to the real physical frequency β^* via the relations $\beta = \beta_0\bar{\delta}/\bar{\delta}_0$ and $\beta_0 = \beta^*(\delta_0 R)/U_j$ and t is the physical time made dimensionless by $(\bar{\delta}R)/U_j$.

The shape function Q. For the shape function Q, linear theory gives us the following eigenvalue equation:

$$\pi'' - 2\bar{u}'(\bar{u}-c)^{-1}\pi' - \alpha^2\bar{T}[\bar{T} - M_j^2(\bar{u}-c)^2]\,\pi = 0, \tag{2.8}$$

which has to be supplemented by homogeneous boundary conditions, to be prescribed later. π is the shape function of the pressure distribution, primes denote differentiation with respect to η, $c = c_r + ic_i$ is the complex phase velocity of the wave and $\alpha = \alpha_r + i\alpha_i$ is the complex wavenumber. Equation (2.8) is written in a dimensionless form in which $\bar{\delta}$, U_j, ρ_j and T_j serve as scales. Locally we have $\beta = \alpha c$. Once the eigenvalue problem has been solved for π, the shape functions of the other components of the disturbance may be obtained from the local linear theory. The varicose and sinuous modes are described by the following boundary conditions imposed on the axis (see Lees & Gold 1964):

$$\pi'(0) = 0 \quad \text{(varicose mode)}, \tag{2.9a}$$

$$\pi(0) = 0 \quad \text{(sinuous mode)}. \tag{2.9b}$$

In the ambient field far away from the jet the radiation condition dictates the other boundary condition for the pressure perturbation. We obtain

$$\left.\begin{aligned} &\pi' + \alpha T_a(1 - M_a^2c^2)^{\frac{1}{2}}\pi = 0 \quad \text{as} \quad \eta \to \infty,\\ \text{with} \quad &\mathscr{R}[\alpha(1-M_a^2c^2)^{\frac{1}{2}}] > 0,\\ &M_a^2 = T_a^{-1}M_j^2, \quad T_a = 1 + \tfrac{1}{2}(\gamma-1)\,M_j^2, \end{aligned}\right\} \tag{2.10}$$

where M_a is the Mach number based on the jet exit velocity and the ambient speed of sound. Note that $\alpha_i < 0$ for spatially amplifying waves. The behaviour of a neutral wave ($\alpha_i = 0$, $c_i = 0$) in the ambient field depends on whether or not the wave is supersonic with respect to the ambient speed of sound. When the wave is supersonic ($M_a c_r > 1$) we obtain a laterally non-decaying harmonic solution similar to the Mach waves generated by a supersonic flow over a wavy wall. When the wave is subsonic ($M_a c_r < 1$) the solution decays exponentially. When c is complex the wave always decays exponentially because c_i introduces an imaginary part into $(1-M_a^2 c^2)^{\frac{1}{2}}$. However, we shall see that a supersonic amplifying wave does not induce the same near field as a subsonic amplifying wave. It should be pointed out that a wave which is locally supersonic in the jet region may be subsonic with respect to the ambient field. The semicircle theorem (Drazin & Howard 1966) limits the range of c; consequently no supersonic waves can exist beyond the mean flow sonic point $U_{\mathbb{C}}^2 M_a^2 = 1$. So far our remarks have been concerned about the shape function Q in (2.8).

The amplitude function A. The amplification and decay of the large-scale structure in a developing mean flow comes primarily from the amplitude function $A(x)$, and this is determined from the energy balancing mechanisms of (2.6) after substitution of (2.7) rather than from the amplification rates of the local linear theory. The linear eigenfunctions Q and the associated amplification rates play a subsidiary role in that they occur in integrals associated with the physical mechanisms of energy production, pressure work and turbulent 'dissipation' in (2.6). Such integrals appear as x-dependent coefficients in the equation for $A(x)$. These are discussed in more detail in Liu (1974).

With regard to the integrands of these interaction integrals, the symmetry of the mean flow about the centre-line of the jet bears directly on such eigensolutions of (2.8). Two fundamental modes of disturbance exist: varicose and sinuous. (The symmetry inherent in our analysis is not present in Liu's mixing-layer analysis.) Some information about the relative importance of these two modes exists in the literature for a non-developing parallel mean flow. Lessen, Fox & Zien (1965) considered a compressible top-hat plane jet profile with a time-like amplifying disturbance. According to their calculations the sinuous mode is more unstable than the varicose mode. Mattingly & Criminale (1971) considered an incompressible fully developed plane jet with a spatially amplifying disturbance. Again, their analysis predicts a dominating sinuous disturbance. However, the two idealized cases mentioned do not apply to a real developing jet. In this paper we investigate the development of the two fundamental modes of the disturbance, taking into account the spread of the initial mixing regions and their merging downstream. That is, the two modes for Q which occur under the interaction integrals are used to study the streamwise development of $A(x)$, subject to the 'spectrum' of initial conditions A_0.

The near jet noise field

It has been shown (Liu 1974) that, because the Q shape functions of (2.7) decay laterally (radially in the case of a round jet) in a weakly exponential manner, the instability wave influences the 'near field' well beyond the confines of the jet, and this is supported by near-field observations (see, for instance, Lassiter & Hubbard 1956; Howes *et al.* 1957). Such striking agreement is obtained through performing *direct* calculation of near-field properties from large-scale quantities in the form of (2.7), rather than performing a retarded potential calculation. The present work is intended to consider contributions to such 'near-field' properties from the sinuous and varicose modes. In so doing, we obtain some understanding of the behaviour of such aerodynamic sound sources. The far sound field, which is not considered here, must then be obtained through a retarded potential calculation (Lighthill 1952, 1962) that includes the source contributions.

The direct calculation of the near-field properties in terms of an averaged energy flux or the square of the pressure fluctuations would contain, from the form of (2.7), the square $|A(x)|^2$ of the amplitude function and products of the shape function Q^2 which describes the local lateral behaviour according to the local characteristics of the wave. Any properly defined shape function should give us the desired qualitative lateral behaviour but for definiteness we use the pressure–velocity correlation vector. Consequently, we define a local intensity vector (which should not be confused with that obtained by a retarded potential consideration) as

$$\mathbf{I} = \overline{p'u'}\mathbf{i} + \overline{p'v'}\mathbf{j}, \tag{2.11}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions, respectively. Upon expressing the components of $\mathbf{I}$ in terms of the shape functions we find that in the ambient field

$$\left.\begin{aligned}\overline{v'p'} &\sim 2|A|^2 \exp\{-2y\mathscr{R}[\alpha(1-M_a^2c^2)^{\frac{1}{2}}]\}\, T_a \mathscr{R}[i\alpha(1-M_a^2c^2)^{\frac{1}{2}}]/\beta,\\ \overline{u'p'} &\sim 2|A|^2 \exp\{-2y\mathscr{R}[\alpha(1-M_a^2c^2)^{\frac{1}{2}}]\}\, T_a \mathscr{R}(c^{-1}),\end{aligned}\right\} \tag{2.12}$$

where $|A|^2$ is the square of the amplitude of the wave determined by (2.6).

Within the framework of our analysis the eigenvalue problem gives us an approximate description of the various wave shape functions. However, the extent in the streamwise direction of significant wave activity is determined primarily by $|A|^2$, whose evolution is governed by (2.6). In other words, the energetics of the flow in the jet determine the natural streamwise cut-off of the wave.

3. Results and discussion

In our numerical example we used Eggers' data (1966) for his $M_j = 2{\cdot}22$ jet and estimated the initial boundary-layer thickness as one-tenth of the exit radius. The lengths of the potential core and the sonic core obtained in the calculations are in good agreement with experimental evidence (Merkine 1974). In order to integrate (2.6) the initial amplitude of the wave together with its physical frequency must be specified, and we chose a broad frequency band with $|A|_0^2 = 10^{-5}$ as the initial value of the square of the wave amplitude as already discussed. This choice is of the right order of magnitude for the 'naturally' existing disturbances in the flow field (Liu 1974). Our frequency range corresponds to a range of the frequency parameter β_0 of 0·01–0·1, which in turn corresponds to a dimensional frequency range of 0·1–10 kHz. The noise frequency spectrum obtained by Jones (1971) for a jet similar to Eggers' indicates that our range of frequencies covers most of the spectrum. The solution of (2.6) has justified the decoupling of this equation from the rest of the mean flow equations.

The development of the amplitude of the wave and consequently the wave-induced noise sources depends on the role played by the various energy exchange mechanisms appearing in (2.6). In all the calculations performed it was found that, in the early stages of the wave development, 'production' dominates pressure work and 'turbulent dissipation' and the amplitude of the wave increases rapidly. At more advanced stages pressure work and 'turbulent dissipation' become comparable to the 'production' term and eventually override it. This behaviour causes the amplitude of the wave to attain a peak and eventually to decay. It has also been found that, the lower the frequency, the further downstream the peak is located. Observations in the near field (Lassiter & Hubbard 1956; Howe *et al.* 1957) indicate that high frequency contributions to the pressure fluctuations or intensity dominate in the region near the jet exit, whereas low frequency contributions dominate in the region far downstream. This is in accordance with our purely local considerations. In these earlier near-field observations, the contributions from the sinuous or varicose modes are not differentiated. Our results for these two modes therefore provide such an understanding. Our discussions thus far, then, suggest methods for noise source suppression according to which jet noise control can be achieved by controlling the mechanisms governing the development of the amplitude of large-scale waves. Liu (1974) gives a rather extensive treatment of the subject. Our results for the development of the wave amplitude and the various energy exchange mechanisms are entirely similar to his, and therefore will not be represented in detail here. Instead, we shall elucidate the roles played by the sinuous and varicose modes.

Figures 2 (*a*) and (*b*) depict the behaviour of the real part of the complex phase velocity of the wave for the sinuous and varicose modes of the disturbance, respectively. For clarity, in this and in subsequent figures, c_R, x, y and α_i are dimensional. For the sinuous mode the wave starts off with a subsonic velocity. If the frequency of the wave is high enough it accelerates and saturates about a supersonic speed which is higher for higher frequencies. The behaviour is different for low frequencies. When $\beta_0 = 0{\cdot}01$ the wave reaches the developed region with subsonic speed, then following an adjustment region, its phase velocity begins to decay as it is limited by the decrease in $U_{\mathbb{C}}$. It will be shown later that supersonic waves influence the near field more profoundly than subsonic waves. The varicose mode (figure 2*b*) shows different behaviour from the sinuous mode. We find that low frequencies are associated with high phase velocities. For $\beta_0 = 0{\cdot}01$ the wave starts off with a supersonic phase velocity. For $\beta_0 = 0{\cdot}075$ the behaviour is similar to the sinuous case except that the saturated value is attained sooner.

Our results indicate that we are dealing with large-scale instability waves, since the wavelengths are of the same order of magnitude as the jet diameter. We have also found that the local linear theory predicts that the sinuous waves have larger local amplification rates $-\alpha_i R$ than the varicose waves, as has already been suggested by earlier work. For later reference, we show the linear local amplification rates for the two modes in figure 3. As we have already dis-

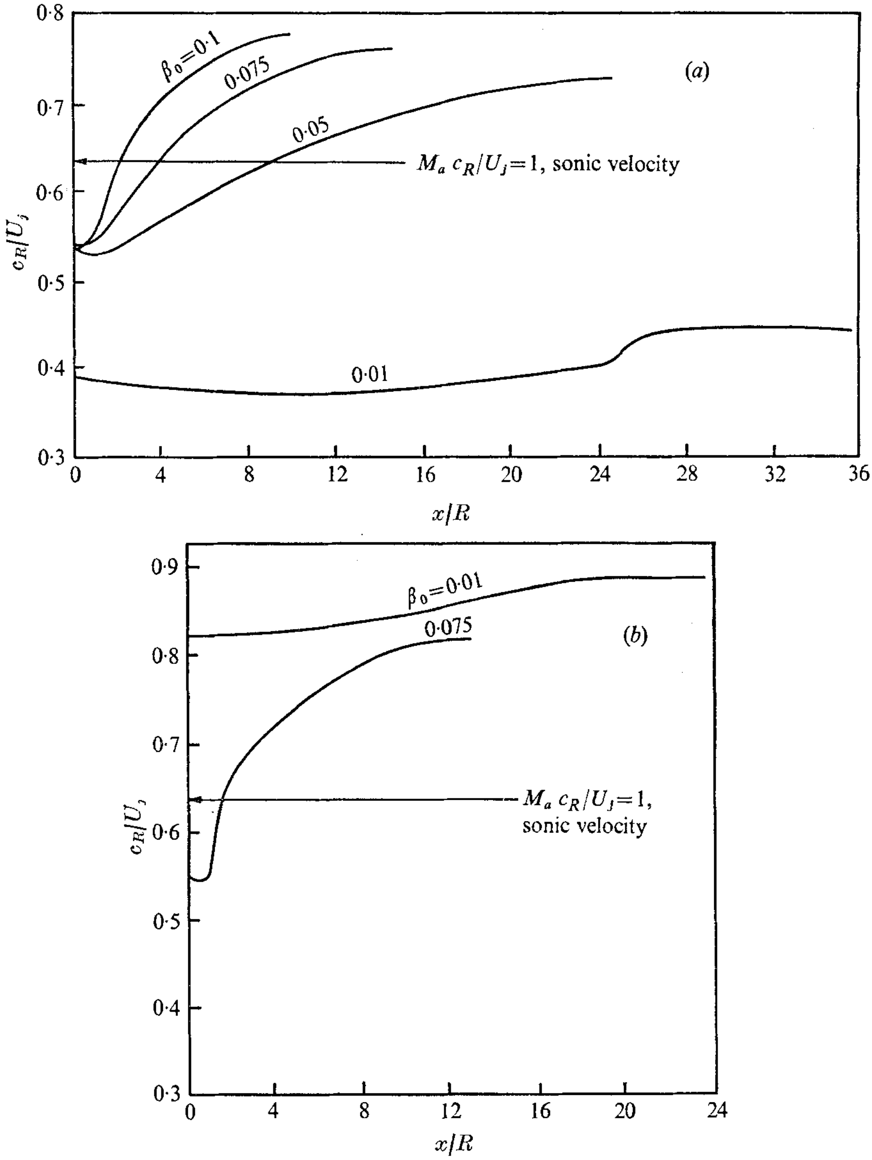


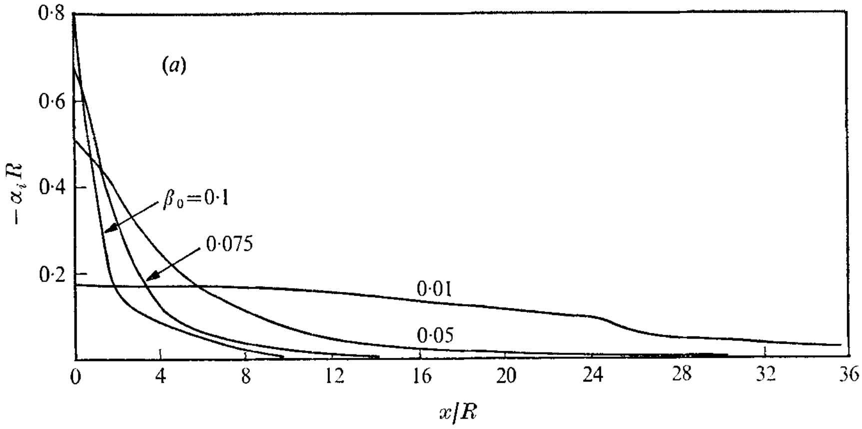
FIGURE 2. Streamwise development of the phase velocity at $M_j = 2{\cdot}22$. (*a*) Sinuous mode. (*b*) Varicose mode. c_R and x are dimensional.

cussed, the eigenfunctions of the linear theory as well as $-\alpha_i$ provide the vertical structure while the amplitude function gives streamwise structure according to (2.6).

Figures 4 and 6 depict levels of constant normal intensity ($\overline{v'p'} =$ constant) expressed in decibels. In these units the wave-induced normal intensity flux is given by

$$I_N = 10 \log \frac{\overline{v'p'}\, U_j^3 \rho_j}{I_{\text{ref}}}\,\text{db}, \quad I_{\text{ref}} = 10^{-12}\,\text{W/m}^2. \tag{3.1}$$

For Eggers' jet we have $U_j = 538$ m/s and $\rho_j = 2{\cdot}404$ kg/m^2.



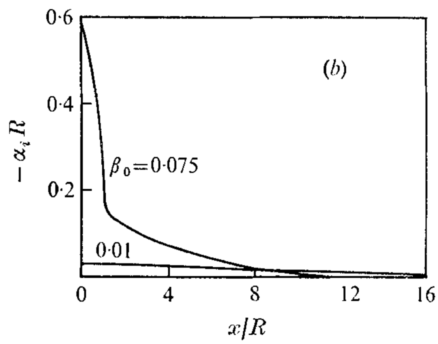
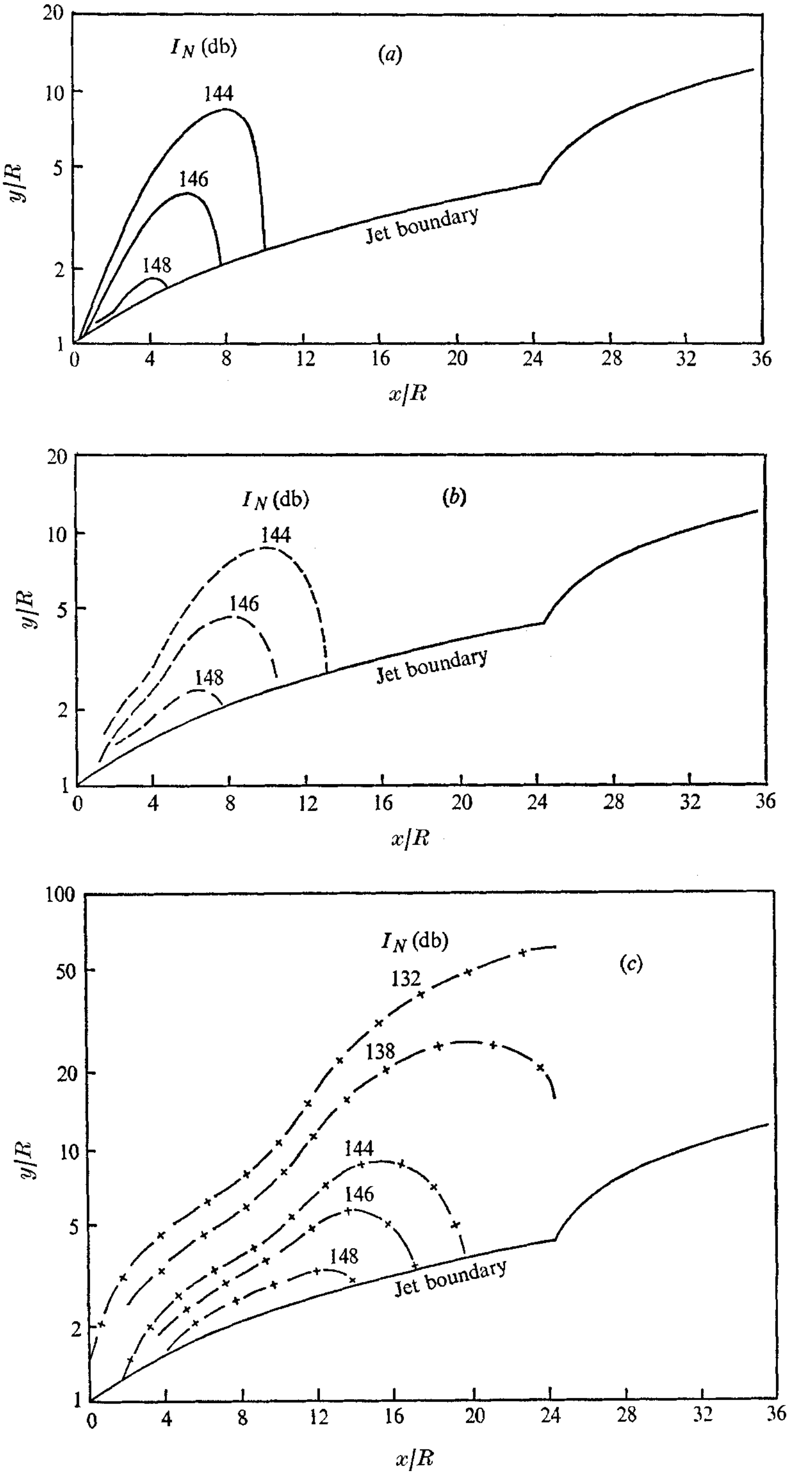


FIGURE 3. Streamwise development of the local linear amplification rates $-\alpha_i R$ at $M_j = 2{\cdot}22$. (a) Sinuous mode. (b) Varicose mode. α_i and x are dimensional.

Acoustic measurements are made by determining the pressure field. Our results are represented in terms of the wave-induced normal intensity $\overline{v'p'}$, but the local linear theory provides us with a proper conversion relation through the form given by (2.7). We find that in the ambient field

$$\overline{p'^2} = \overline{v'p'}\,\frac{|c|^2}{c_r T_a}\,\frac{\alpha_r}{\mathscr{R}[i\alpha(1-M_a^2c^2)^{\frac{1}{2}}]} \tag{3.2}$$

and we refer, of course, to local contributions only. Figure 4 represents the wave-induced normal intensity for a range of frequencies for the sinuous mode of the disturbance. It is clear that high frequency waves peak earlier than low frequency waves. This result is also supported by Liu's work and is a dominating feature of the experimental observations. An important feature is that for a fixed y station the dominant intensity shifts downstream with the highest intensity occurring at the end of the potential core. In the subsequent fully developed region, where the fine-scale turbulence is more active than in the mixing region, the amplitude of the wave decays rapidly as a result of the enhanced 'dissipation' of its kinetic energy. This is reflected in figure 4 in the rapid decay of the normal intensity for



FIGURES 4 (a–c). For legend see next page.

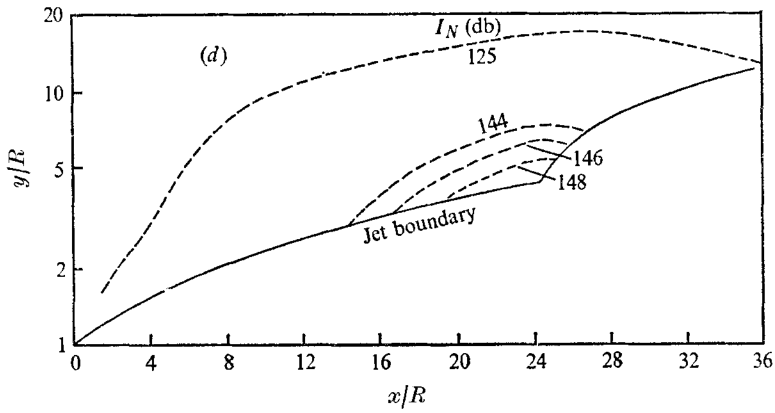


FIGURE 4. Contours of the wave normal intensity level I_N (relative to 10^{-12} W/m²) for $M_j = 2{\cdot}22$ for the sinuous mode at various values of the frequency parameters. (a) $\beta_0 = 0{\cdot}1$. (b) $\beta_0 = 0{\cdot}075$. (c) $\beta_0 = 0{\cdot}05$. (d) $\beta_0 = 0{\cdot}01$. x and y are dimensional.

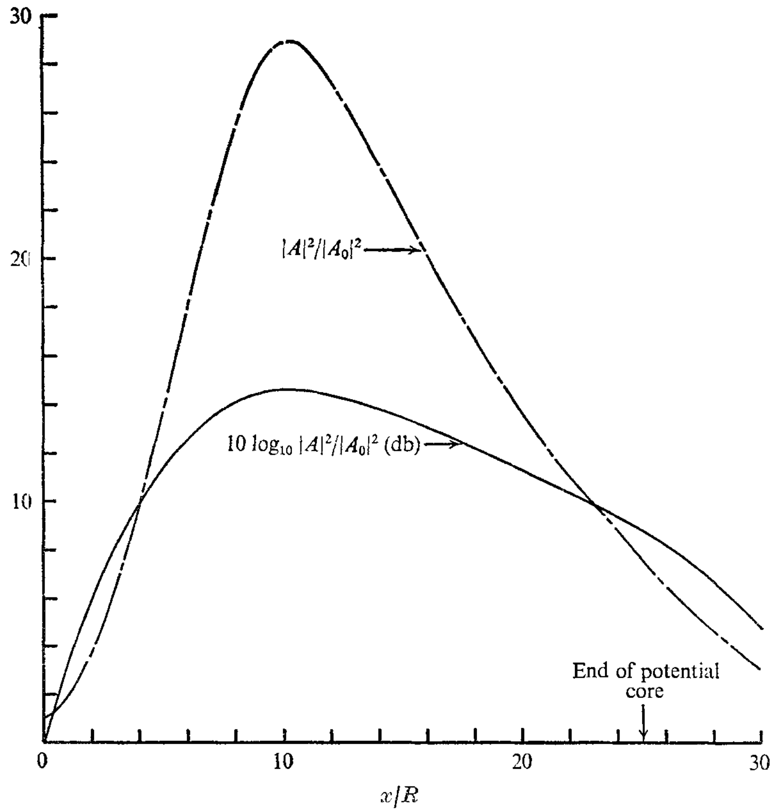


FIGURE 5. Streamwise development of the amplitude function for $M_j = 2{\cdot}22$, $|A_0|^2 = 10^{-5}$, $\beta_0 = 0{\cdot}05$, sinuous mode. x is dimensional.

all frequencies and it appears to explain the appearance of the observed maximum acoustic intensity in the vicinity of the end of the potential core (see, for example, Potter & Jones 1968; Bishop *et al.* 1971).

We should point out that *only* the non-parallel formulation for $|A(x)|^2$ which includes the proper energy exchange mechanisms accounts for the decay in the amplitude of the wave. The local linear theory cannot account for this decay since it indicates the existence of a nearly neutral wave far downstream (see figure 3). The linear theory, of course, provides local lateral shape functions in our consideration of the developing mean flow (Liu 1974). As an illustration, the development of $|A|^2/|A_0|^2$ for the $\beta_0 = 0{\cdot}05$ sinuous mode is shown in figure 5; it makes a maximum contribution of about 15 db. The streamwise development of I_N (figure 4c) essentially follows that of $10\log_{10}|A|^2/|A_0|^2$.

The fast decay of the flow field downstream of the potential core restricts the phase velocity of the large-scale eddies which enter the developed region, since the phase velocity can never exceed the local maximum flow velocity and therefore no supersonic phase velocities can exist downstream of the sonic point. Our results indicate that supersonic wavelike eddies attenuate rapidly in the fully developed region. This result is supported by Salant, Gregory & Kolesar (1971), who did not observe ambient waves downstream of the tip of the potential core. The $\beta_0 = 0{\cdot}01$ wave, though subsonic throughout its development, depicts the narrow lateral extent of the region of sources generated by the subsonic wavelike eddies that can exist downstream of the sonic point. The observations which indicate that the intensity decays rapidly downstream of the sonic point result from the fact that only subsonic waves can exist in this region. Since the lateral extent of the intensity of supersonic eddies is greater than that of subsonic eddies it might be conjectured that supersonic eddies exert a greater influence over the far field than subsonic eddies and that the main noise-producing eddies occur before the termination of the potential core region.

In figure 6 we show the lines of constant normal intensity of the wavelike eddies for the varicose mode compared with some of the reuslts for the sinuous mode for the same frequency parameter β_0. For the case $\beta_0 = 0{\cdot}01$, we note from the discussion of phase velocities, shown in figure 2, that the varicose mode is supersonic at the outset while the sinuous mode remains subsonic throughout its streamwise development. Also, from the calculated results of the local linear theory, the general level of $-\alpha_i R$ for the varicose mode is much lower. Thus these facts account, through (2.12), for the varicose mode having a much larger influence than the sinuous mode for the low frequency case $\beta_0 = 0{\cdot}01$. For this case, though dominating laterally, the varicose mode has a much shorter streamwise lifetime. For the higher frequency modes, typified by the case $\beta_0 = 0{\cdot}075$, the varicose mode, which also starts out subsonic, becomes supersonic earlier and has a generally lower level of $-\alpha_i R$ than the corresponding sinuous mode. Thus the varicose mode has a greater influence laterally and again has a shorter streamwise lifetime. In general, given the same initial 'natural' (Liu 1974) excitation level of $|A|_0^2 = 10^{-5}$, the varicose-mode intensity levels are relatively lower close to the jet than those for the sinuous mode. Although the fact that the Strouhal number based on the nozzle diameter $2R$ is not necessarily the appropriate

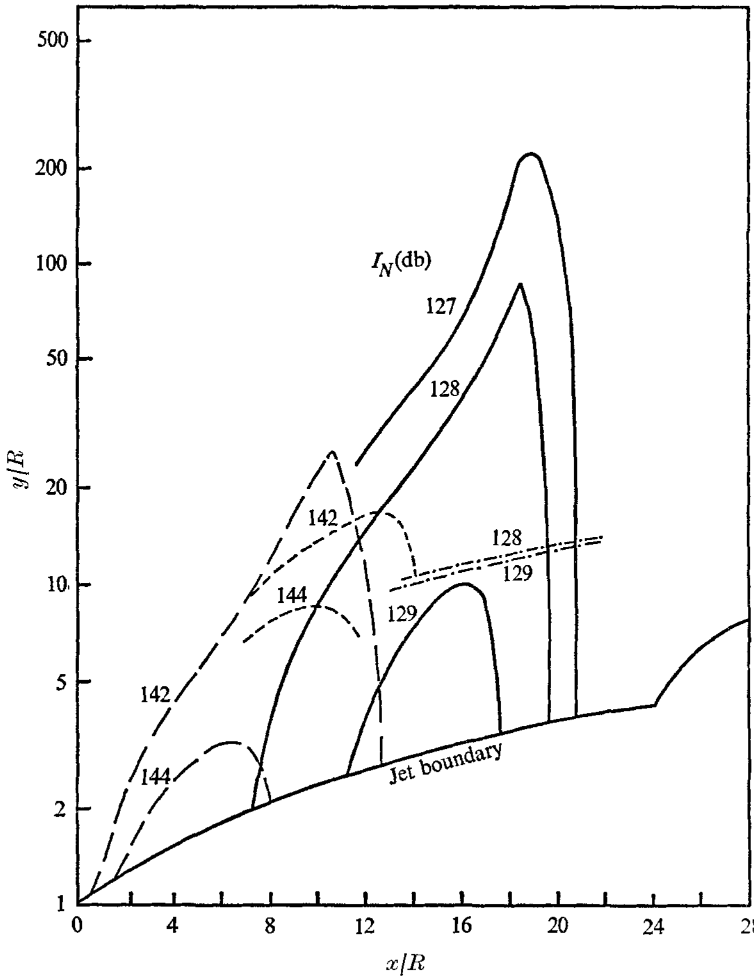


FIGURE 6. Contours of the normal intensity level I_N (relative to 10^{-12} W/m²) for $M_j = 2{\cdot}22$ for the varicose mode; sinuous-mode contours are included for comparison. Varicose mode: ——, $\beta_0 = 0{\cdot}01$; — —, $\beta_0 = 0{\cdot}075$. Sinuous mode: — - —, $\beta_0 = 0{\cdot}01$; - - - -, $\beta_0 = 0{\cdot}075$. x and y are dimensional.

indicator of the 'peak emitter' was discussed in Liu (1974), we note here its correspondence with the appropriate frequency parameter β_0:

$$St_d = \beta_0(2R/\delta_0)\,(\delta_0/\bar{\delta}_0)\,2\pi.$$

For $\beta_0 = 0{\cdot}01$ and $0{\cdot}075$, St_d is $0{\cdot}05$ and $0{\cdot}36$, respectively, for a ratio of the initial boundary-layer thickness to the nozzle radius of $0{\cdot}1$ and for $M_j = 2{\cdot}22$, $\delta_0/\bar{\delta} \simeq 1{\cdot}49$ according to the Howarth transform inversion.

For a low speed plane jet Oseberg & Kline (1971) observed that in the near field a predominant varicose mode existed in the region before the end of the potential core, while the sinuous mode existed further downstream. This is thus in agreement with our discussions.

4. Concluding remarks

It has been the aim of this paper to elucidate the question of the streamwise lifetime of the varicose and sinuous modes of the wavelike eddies in a developing real jet flow. The round-jet problem within the eddy-viscosity framework is now a problem of a computational nature, the significant physical features being exhibited by the present, much simpler plane problem. The problems of understanding the wave-induced turbulent Reynolds stresses and their role in the 'dissipation' of the large-scale structure and the contributions to the far sound field from the large-scale structure are being investigated and will be reported at a later date.

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